

Quantum Deformations of Space-Time Symmetries and Interactions

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Abstract

We discuss quantum deformations of Lie algebra as described by the noncoassociative modification of its coalgebra structure. We consider for simplicity the quantum $D = 1$ Galilei algebra with four generators: energy H , boost B , momentum P and central generator M (mass generator). We describe the nonprimitive coproducts for H and B and show that their noncocommutative and noncoassociative structure is determined by the two-body interaction terms. Further we consider the case of physical Galilei symmetry in three dimensions. Finally we discuss the noninteraction theorem for manifestly covariant two-body systems in the framework of quantum deformations of $D = 4$ Poincaré algebra and a possible way out.

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1 Introduction

In standard formulation of symmetry schemes the infinitesimal symmetries are described by classical Lie algebras \hat{g} . If we consider the classical Lie algebra \hat{g} as a bialgebra, we should provide additional information about the form of the coproduct $\Delta(\hat{g})$. Usually in classical and quantum mechanics we represent \hat{g} in terms of one-particle observables forming the algebra \hat{A} , and the coproduct $\Delta(\hat{g}) \subset \hat{A} \otimes \hat{A}$ is a function of two-particle observables. In particular if \hat{g} describes the group of motions which includes the energy (generator of time translations) the choice

$$\Delta(\hat{g}) = \hat{g} \otimes 1 + 1 \otimes \hat{g} \quad (1.1)$$

means that the two-particle system is a free one, because all the space-time symmetry generators are additive.

One of the aims of this paper is to show that in the category of noncoassociative bialgebras which describe space-time symmetries the information about the form of the coproducts for space-time symmetry generators carries the information about the two-particle interactions. Because due to Drinfeld theorem [1] in the formalism of quantum deformations of Lie algebras it does exist the isomorphism between the classical ($\mathcal{U}(\hat{g})$) and deformed ($\mathcal{U}_q(\hat{g})$) enveloping algebras, it is possible to transfer all the quantum deformation to the coalgebra sector (see e.g. [2]). Our discussion deals with such a case when the Lie algebra is classical, but the coproduct (which we recall is a homomorphism of \hat{g}) is nonadditive and noncocommutative. We shall search for such coproducts as determined by the symmetry-invariant interactions. Our result here consists in writing the deformed coproduct for the generators of space-time symmetries as follows

$$\Delta(\hat{g}) = \Delta^{(0)}(\hat{g}) + \Delta^{(int)}(\hat{g}) \quad (1.2)$$

and relate $\Delta^{(int)}(\hat{g})$ with two-body interactions.

For simplicity we consider as an example the $D = 1$ Galilei algebra. In such a case the algebra A is the one-dimensional Heisenberg algebra with two generators \hat{x} , \hat{p} , and the nonrelativistic space-time algebra takes the form ($\hat{g} = (B, P, H, M)$)

$$[B, P] = iM, \quad (1.3a)$$

$$[B, H] = iP, \quad (1.3b)$$

$$[P, H] = [M, \hat{g}] = 0, \quad (1.3c)$$

where B denotes nonrelativistic boost in two-dimensional space-time (x, t) , P is the momentum (generator of space translations), H is the energy (generator of time translations), and M is the central charge generator describing the mass operator. The Galilei symmetry is described by the following $D = 2$ space-time transformations:

$$\begin{aligned}x' &= x + a + vt, \\t' &= t + b.\end{aligned}\tag{1.4}$$

If M takes the numerical mass value m , the one-particle realization of the algebra (1.3a)-(1.3c) in terms of the canonical variables (\hat{x}, \hat{p}) can be chosen as follows (see e.g. [3]):

$$B = m\hat{x}, \quad P = \hat{p}, \quad H = \frac{\hat{p}^2}{2m}, \quad M = m \cdot 1.\tag{1.5}$$

We see that the relation (1.3a) can be identified with the Heisenberg relation $[\hat{x}, \hat{p}] = i$ (we put $\hbar = 1$).

In the following paragraph we shall determine the coproduct of $\hat{g} = (B, P, H, M)$ by considering the $D = 1$ nonrelativistic Galilei-invariant interacting two-body systems. It should be stressed that our coproducts are not coassociative. We shall comment further on the $D = 3$ Galilean case. In the next paragraph we shall discuss from the point of view of the deformed coproducts the relativistic Poincaré invariant two-body systems. In final section we present the conclusion: that in the Hopf-algebraic (or quantum group) framework one can understand why large class of Galilei-invariant interactions are allowed and why there is valid a no-go theorem concerning four-dimensional covariant two-body interactions.

2 The coproduct as the characterization of the two-body interactions

Let us consider the following two-body Lagrangian describing Galilei-invariant interacting two-particle system in one-dimensional space of the following form ($m > \kappa$).

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}\kappa(\dot{x}_1^2 - \dot{x}_2^2) - \frac{\omega}{2}(x_1 - x_2)^2.\tag{2.1}$$

It is easy to see that the canonical momenta ($i = 1, 2$)

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i, \quad (2.2)$$

where $m_1 = m + \kappa$, $m_2 = m - \kappa$.

The energy operator H takes the form

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{w}{2}(x_1 - x_2)^2, \quad (2.3)$$

where for $i = 1, 2$

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij}. \quad (2.4)$$

The lagrangian (2.1) transforms under the special Galilei transformation ($v \neq 0$ in (1.4)) as follows

$$L \rightarrow L' = L + \frac{d}{dt}F, \quad (2.5)$$

where ($R = \frac{x_1 + x_2}{2}$, $x = x_1 - x_2$)

$$F = 2mRv + mv^2t + (c + \kappa x)v. \quad (2.6)$$

The generator of special Galilei transformations is given by the formula

$$B = \left. \frac{\partial F}{\partial v} \right|_{\substack{v=0 \\ t=0}} = 2mR + (c + \kappa x) = 2m\tilde{R} + c, \quad (2.7)$$

where $\tilde{R} = (2m)^{-1}(m_1x_1 + m_2x_2)$. If we define

$$P = p_1 + p_2, \quad (2.8)$$

one can check that three generators (2.3), (2.7)-(2.8) satisfy the algebra (1.3a)-(1.3b), with $M = 2m$.

Below we shall write these generators in the coproduct form. We shall consider both κ and ω in the Lagrangean (2.1) as describing the interaction terms. Because (see (1.5); $i = 1, 2$):

$$x_i = \frac{1}{m}B_i, \quad P_i = p_i, \quad H_i = \frac{p_i^2}{2m}, \quad M_i = m, \quad (2.9)$$

one can write the two-body generators in the following form (we choose $c = 0$, or shift $B \rightarrow B - c$):

$$B = \frac{m_1}{m}B_1 + \frac{m_2}{m}B_2, \quad (2.10a)$$

$$H = \frac{m}{m_1}H_1 + \frac{m}{m_2}H_2 + \frac{\omega}{2m^2}(B_1 - B_2)^2, \quad (2.10b)$$

$$P = P_1 + P_2, \quad M = M_1 + M_2, \quad (2.10c)$$

or using the language of coproduct we should write

$$\Delta(B) = \frac{m_1}{m}B \otimes 1 + \frac{m_2}{m}1 \otimes B, \quad (2.11a)$$

$$\begin{aligned} \Delta(H) &= \frac{m}{m_1}H \otimes 1 + \frac{m}{m_2}H \otimes 1 \\ &\quad + \frac{\omega}{2m^2}(B^2 \otimes 1 - 2B \otimes B + 1 \otimes B^2), \end{aligned} \quad (2.11b)$$

$$\Delta(P) = P \otimes 1 + 1 \otimes P, \quad \Delta(M) = M \otimes 1 + 1 \otimes M. \quad (2.11c)$$

It is easy to see that the coproducts (2.11a)-(2.11c) are the homomorphism of classical $D = 1$ Galilei algebra provided the central mass generator M is diagonal, i.e. $M = m \cdot 1$.

We would like to make the following comments:

- i) The space-dependent harmonic potential modifies the primitive coproducts for the energy generator. Here we show that if the interaction modifies kinematic term by introducing the split of masses ($\kappa \neq 0$), then we obtain the nonadditivity of the boost generators.
- ii) It appears that the coproduct for B can be made quite complicated by replacing in (2.1) the “mass-difference” kinematic term by the following general velocity-dependent interaction ¹

$$\frac{\kappa}{2}(\dot{x}_1^2 - \dot{x}_2^2) \rightarrow \kappa(x)\dot{x}_i\dot{R}_i, \quad (2.12)$$

where $\dot{x}_i\dot{R}_i = \frac{1}{2}(\dot{x}_1^2 - \dot{x}_2^2)$. We see that the choice $\kappa(x) = \kappa$ corresponds to the case considered above. For the choice of general function $\kappa(x)$ one obtains after the substitution (2.12) the formulae (2.3) with $m_1 \rightarrow$

¹Such a term has been discussed very recently by J. Łopuszański and P. C. Stichel [9].

$m_1(x) = m + \kappa(x)$, $m_2 \rightarrow m_2(x) = m - \kappa(x)$. Because in the coalgebra language one can write

$$x = x_1 - x_2 \equiv \frac{B}{m} \otimes 1 - 1 \otimes \frac{B}{m}, \quad (2.13)$$

the formula (2.10a) after the substitution (2.12) can be generalized in the straightforward way. Further the harmonic potential can be replaced in (2.1) with general velocity-dependent potential

$$\frac{\omega^2}{2} x^2 \rightarrow U(x, \dot{x}) \quad (2.14)$$

The modification (2.14) of (2.1) affects only the Hamiltonian (2.3), in which the potential energy is replaced with momentum-dependent potential

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x, p), \quad (2.15)$$

where $p = (2m)^{-1}(m_2 p_1 - m_1 p_2)$. Using the substitution (2.14) and

$$p \rightarrow \frac{1}{2m}(m_2 \cdot P \otimes 1 - m_1 \cdot 1 \otimes P) \quad (2.16)$$

one can rewrite (2.15) as the nontrivial coproduct for H .

- iii) In all considered cases, also after the substitutions (2.12) and (2.14), the momentum variable remains additive, i.e. the primitive coproduct (2.10a) for P remains valid in the presence of arbitrary Galilei-invariant interactions.
- iv) It is interesting that for the coproducts (2.11a)–(2.11c) the coassociativity condition

$$(\Delta \otimes 1)\Delta(\hat{g}) = (1 \otimes \Delta)\Delta(\hat{g}), \quad \hat{g} = H, B, P, M \quad (2.17)$$

is not valid for any choice of the two-body potential $V(x, p)$ (in particular this statement can be checked for the choice of coproducts (2.11a)–(2.11c)). It appears that the coassociativity imposes very stringent restrictions on the class of allowed interactions which if are possible should have very unconventional form. It should be mentioned that the violation of coassociative structure in quantum mechanics has been recently discussed in [10].

- v) The present discussion can be also extended to the case of $D = 3$ Galilean system. The realization (1.5) is generalized in straightforward way with \hat{x} and \hat{p} replaced by the three-vectors, and quantum-mechanical realization $M_i = \epsilon_{ijk} \hat{x}_j \hat{p}_k$ for the angular-momenta. For the case with spin the one-particle realization of $D = 3$ Galilei algebra takes the form

$$\begin{aligned} M_i &= \epsilon_{ijk} \hat{x}_j \hat{p}_k + \hat{S}_i, & B_i &= m \hat{x}_i \\ P_i &= p_i, & H_i &= \frac{\hat{p}_i \hat{p}_i}{2m}, & M_i &= m, \end{aligned} \quad (2.18)$$

i.e. the 8-dimensional phase space $(\hat{x}_i, \hat{p}_i, \hat{S}_i)$, where S_i^2 is a Casimir, can be reexpressed in terms of $D = 3$ Galilei generators. In particular any interaction $U(\vec{x}, \vec{p}, \vec{x}\vec{S}, \vec{p}\vec{S})$ (where \vec{x} is relative coordinate, \vec{p} is relative momentum and \vec{S} is the total spin) in the Hamiltonian can be described by the correction to the primitive coproduct for the energy operator H .

3 Relativistic interactions and no-interaction theorem

It is well-known that in comparison with Poincaré invariance the Galilean invariance is much less restrictive for the construction of invariant two-body interactions. There are possible the following two approaches to relativistic Poincaré-invariant interactions:

- i) Let us consider for simplicity spinless relativistic particles. One assumes that the relativistic system is described by eight-dimensional phase-space $(\hat{x}_\mu, \hat{p}_\mu)$ where

$$[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}, \quad (3.1)$$

with some additional constraints imposed. In such a case e.g. two-body potentials would be described by the function $V(x_\mu)$, where $x_\mu = x_\mu^{(2)} - x_\mu^{(1)}$. Only the space coordinates x_μ ($\mu = 1, 2$) can be expressed with the Poincaré algebra generators M_i, N_i, P_i, P_0 (where M_i describe space rotations, N_i – three relativistic boosts and $P_\mu = (P_i, P_0)$ – the

fourmomenta) by the following formula

$$X_i = P_0^{-\frac{1}{2}} N_i P_0^{-\frac{1}{2}} , \quad (3.2)$$

Because the two-body covariant potential $V(x_\mu)$ depends also on relative time variable, it can not be reexpressed in terms of a pair of Poincaré generators describing two relativistic particles. Concluding it is not possible to describe the energy of covariantly interacting two-particle system as a coproduct of the Poincaré algebra generator P_0 .

- ii) One can also impose the conditions of relativistic invariance by considering only the three-dimensional coordinates and momenta $x_i^{(k)}, p_i^{(k)}$ (for two-body systems $i = 1, 2$). The Poincaré generators describing relativistic interacting 2-particle system should satisfy the following conditions [3-6]:

$$\begin{aligned} [M_i, x_j^{(k)}] &= i\epsilon_{ijk} x_j^{(k)} , \\ [L_i, x_j^{(k)}] &= i x_i^{(k)} [H, x^{(k)}] \end{aligned} \quad (3.3)$$

with

$$P_i = p_i^{(1)} + p_i^{(2)} \quad (3.4)$$

Let us assume that the two-body Poincaré symmetry generators satisfy the classical Poincaré algebra. In such a framework the problem of the existence of covariant interaction has been studied very extensively [4-6]. Under the assumption that the Hamiltonian is of standard type:

$$\det \left| \frac{\partial^2 H}{\partial p_i^{(k)} \partial p_j^{(l)}} \right| \neq 0 , \quad (3.5)$$

it has been shown [4] that the two-body Poincaré generators are the sum of two free-particle realizations:

$$\begin{aligned} M_i &= \frac{1}{2} \epsilon_{ijk} x_j p_k + S_i , & P_i &= p_i , & P_0 &= (\vec{p}^2 + m^2)^{\frac{1}{2}} , \\ N_i &= (\vec{p}^2 + m^2)^{\frac{1}{2}} x_i + [m + (\vec{p}^2 + m^2)^{\frac{1}{2}}]^{-\frac{1}{2}} \epsilon_{ijk} p_j S_k , \end{aligned} \quad (3.6)$$

i.e. the manifestly covariant relativistic two-body interaction is not allowed.

We see therefore that the relativistic two-body interaction (under some technical assumptions see (3.6)) is not allowed. In the language of quantum deformations of the Poincaré algebra it appears that there does not exist a nontrivial coproduct for classical Poincaré algebra generators satisfying the condition (3.4), i.e.

$$\Delta(P_i) = P_i \otimes 1 + 1 \otimes P_i. \quad (3.7)$$

Indeed, the same conclusion is obtained from the consideration of the κ -deformed Poincaré algebra [7, 8] in the classical algebra basis [2]. It follows from these considerations that the coproduct relation (3.7) has to be modified.

4 Final Remarks

The aim of this paper is to relate the symmetry-invariant interactions with the existence of the deformation of the coproducts for classical symmetry algebras. It appears that if we do not require coassociativity the nonrelativistic Galilean symmetry is not so restrictive — indeed it is possible to find very large class of the deformed coproducts satisfying classical Galilei algebra relations. Quite different situations occurs for relativistic systems. In the framework of standard manifestly covariant relativistic interaction satisfying relations (3.4)-(3.5) the two-body relativistic potential is not allowed. However we know (see e.g. [2]) that there do exist the deformations of coproducts for classical Poincaré algebra, with nonadditive three-momentum generators i.e. with the relation (3.7) modified. We conjecture that such deformed Poincaré-Hopf algebras describe permitted class of the relativistic interactions, however in new category of nonstandard Hamiltonians. These Hamiltonians will have nonlocal structure in time, with modified kinetic terms containing derivatives of arbitrarily high order. The explicit forms of such nonstandard manifestly covariant relativistic interactions are now under consideration.

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